Almost Homogeneous Poisson Spaces [*†]

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Abstract

We prove that any holomorphic Poisson manifold has an open symplectic leaf which is a pseudo-Kähler submanifold, and we define an obstruction to study the equivariance of momentum map for tangential Poisson action. Some properties of almost homogeneous Poisson manifolds are studied and we show that any compact symplectic Poisson homogeneous space is a torus bundle over a dressing orbit.

§1. Introduction

Motivated by the study of Poisson nature of "dressing transformation" for soliton equations, Lu and Weinstein introduced the notion of Poisson action [STS, Lu, LW, W]. Poisson actions are used to understand the "hidden symmetries" of certain integrable systems. When the Poisson structure of the Poisson Lie group is zero, the Poisson structure of the acted Poisson manifold is preserved. So Poisson actions are the generalizations of symplectic actions and Hamiltonian actions. Tangential Poisson action $G \times P \longrightarrow P$ is a special kind of Poisson action which is the most adjacent generalization of symplectic action. The orbits of tangential Poisson action lie in the symplectic leaves of P and the restricted action of G to every symplectic leaf is also a Poisson action (cf. [Y]).

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Momentum mapping for symplectic action on symplectic manifold is the abstract of momentum and angular momentum in classical mechanics.The symmetry of the phase space of a mechanical system leads to some conservative quantities, to reduce the symmetry by these conservative quantities is a way to simplify the mechanical system. So the momentum map plays an important role in reduction. In [Lu], Lu generalized this concept to the tangential Poisson action and studied the Meyer-Marsden-Weinstein momentum reduction for tangential Poisson action on symplectic manifold in regular case. In [Y], we generalized momentum reduction to singular value on any Poisson manifold. In [Gi], Ginzburg studied the further properties of momentum map for tangential Poisson action.

In this paper, we discuss other properties of the momentum mapping for tangential Poisson actions. In sect. 2, we give a detailed description of symplectic stratification of a Poisson manifold. In sect. 3, we in principle classify Poisson Lie structures on abelian Lie groups. In sect. 4, we give the necessary and sufficient condition when a Poisson action is Poisson structure preserved, and define an obstruction to describe the equivariance of momentum map. In sect. 5, we prove that the compact symplectic Poisson homogeneous space is a torus bundle over some dressing orbit, and give an example of non-transitive tangential Poisson action which is not a dressing action.

§2. Poisson manifolds and their symplectic stratifications

A commutative associative algebra $\mathcal{A}$ over field $\mathbb{F}$ is called a *Poisson algebra* if it is equipped with a $\mathbb{F}$-bilinear operation (called *Poisson bracket*) $\{,\} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ satisfying the following conditions:

(i) $(\mathcal{A}, \{,\})$ is a Lie algebra over $\mathbb{F}$;

(ii) the commutative algebraic structure is compatible with the Lie algebraic structure in the following way, *i.e.* , the Leibniz rule is satisfied:

$$\{ab, c\} = a\{b, c\} + \{a, c\}b.$$

An algebraic $\mathbb{F}$-variety V is called an *algebraic Poisson $\mathbb{F}$-variety* if the sheaf $\mathcal{O}_V$ of regular functions carries a structure of a sheaf of Poisson algebra. A complex space W is called a Poisson space if the sheaf of regular functions is a holomorphic Poisson algebra, in the smooth case W is called a *holomorphic Poisson manifold.* A smooth real (resp. complex) manifold P is called a

Poisson manifold if the algebra of smooth functions $\mathcal{A} = C^\infty(M, \mathbb{R})$ (resp. $C^\infty(M, \mathbb{C})$) is equipped with a structure of Poisson algebra over $\mathbb{R}$ (resp. $\mathbb{C}$).

Let P be a Poisson manifold of dimension n. The Leibniz rule of $C^\infty(P)$ implies that the Poisson bracket $\{,\}$ is a derivation in each argument, *i.e.* , $\{f, \cdot\} \in Der(C^\infty(M))$ is a vector field on P. By the skew-symmetry of $\{,\}$, we can conclude that there exists a bi-vector $\pi_P \in \Gamma(\bigwedge^2 TP)$ such that $X_f := \{f, \cdot\}$ and $\{f, g\} = \langle \pi_P^\sharp(df), dg\rangle$, here $\pi_P^\sharp$ is the bundle map defined by π_P. The Jacobi identity just corresponds to $[\pi_P, \pi_P] = 0$, here $[,]$ is the *Schouten bracket* operation (cf. [V, Chapter 1]). If (P, π_P, J) is a holomorphic Poisson manifold with complex structure J, clearly we have $\pi_P \in \Gamma(\bigwedge^2 T^{1,0}P)$, here $T^{1,0}P$ is the holomorphic tangential space defined by J. A Poisson structure on a manifold P defines a Lie algebroid structure on T^*P. The Lie bracket on the space $\Gamma(T^*P)$ of 1-forms is given by

$$\{\alpha, \beta\} = d\pi_P(\alpha, \beta) - \langle \pi_P^\sharp(\alpha), d\beta\rangle + \langle \pi_P^\sharp(\beta), d\alpha\rangle. \tag{2.1}$$

If $V \in \Gamma(TP)$ is a vector field, then

$$\langle V, \{\alpha, \beta\}\rangle = (L_V\pi_P)(\alpha, \beta) - \langle \pi_P^\sharp(\alpha), d(i(V)\beta)\rangle + \langle \pi_P^\sharp(\beta), d(i(V)\alpha)\rangle. \tag{2.2}$$

If π_P is a nondegenerate tensor, the inverse bundle map $(\pi_P^\sharp)^{-1}$ thought of as a two form is a symplectic form, thus P is the unique symplectic leaf. However in general case, π_P may have varying ranks. Fix a point $p \in P$, define

$$\mathcal{C} = \{V \in T_pP | f \in C^\infty(P), X_f(p) = V\}.$$

$\mathcal{C}$ is a differential distribution on P. By Jacobi identity we have $[X_f, X_g] = X_{\{f,g\}}$. So $\mathcal{C}$ is an involutive distribution. In fact, $\mathcal{C}$ is a completely integrable distribution (cf.[V, Theorem 2.6]). We denote the integral leaf through p by $\mathcal{F}_p$. Since Hamiltonian flows defined by Hamiltonian vector fields preserve the Poisson brackets, the restriction of π_P to $\mathcal{F}_p$ has constant rank. In this way we know that $\mathcal{F}_p$ is a symplectic submanifold of P and hence a symplectic leaf of P. If P is compact, let Ham(P) $\subset$ Diff(P) be the Hamiltonian diffeomorphism group with Lie algebra $\{X_f \mid f \in C^\infty(P)\}$. Then the symplectic leaves are just the connected components of Ham(P)-orbits in P.

Let $r(p) := \dim\mathcal{C}_p$, then $r(p)$ is equal to the rank of $\pi_P(p)$. Since π_P is skew symmetric, r is a even integer-valued function bounded by the dimension of P. Choose a local coordinate chart

$U \ni p = (x_1, \cdots, x_n)$ around p. We can write under this coordinate that $\pi_P(p) = \pi_P^{ij}(p)\partial_{x_i} \bigwedge \partial_{x_j}$. Fix a positive integer $k \leqslant n$, for vectors $\overrightarrow{\rho} = (1 \leq \rho_1 < \rho_2 < \cdots < \rho_k \leq n)$ and $\overrightarrow{\delta} = (1 \leq \delta_1 < \delta_2 < \cdots < \delta_k \leq n)$, let $\pi_P^{\overrightarrow{\rho}\,\overrightarrow{\delta}}$ be k-order sub-matrix of π_P whose entries, lines indexed by $\overrightarrow{\rho}$ and columns indexed by $\overrightarrow{\delta}$. Define

$$r_k(p) = \sum_{|\overrightarrow{\rho}|,|\overrightarrow{\delta}|=k} |\ \mathrm{det}\pi_P^{\overrightarrow{\rho}\,\overrightarrow{\delta}}(p)\ |^2, \quad k = 1, 2, \cdots, n,$$

here $|\overrightarrow{\rho}| = k$ means $\overrightarrow{\rho}$ is a vector with n-components. Let $P_l = \{x \in P \mid r(x) = l\}$ and $Q_l = \{x \in P \mid r(x) \leq l\}$. Denote 'max' the maximum of $r(x)$. Then

$$P = \bigcup_{l=1}^{[n/2]} P_l = Q_{\max}.$$

Proposition 1.1 If P is an holomorphic Poisson manifold, then for any positive integer less than $n/2$, P_{2l-1} is empty; if Q_{2l} is non-empty, then the real codimension of the subset Q_{2l-2} in Q_{2l} is of at least 2.

Proof. Clearly P_{2l-1} is empty since the rank of a skew symmetric matrix is always even. We have

$$Q_{2l} = \bigcup_{k=1}^{l} P_{2k} = \{x \in P \mid r_{2k+1}(x) = r_{2k+2}(x) = \cdots = r_n(x) = 0\} \tag{2.3}$$

and

$$Q_{2l-2} = \{x \in Q_{2l} \mid r_{2k}(x) = r_{2k-1}(x) = 0\} \qquad (2.4).$$

Note that all Q_{2k} are analytic subset of P. Since Q_{2l} is non-empty, r_{2l} is a nonzero holomorphic function on P. So the real codimension of Q_{2l-2} in Q_{2l} is of at least 2. □

Corollary 1.1 For holomorphic Poisson manifold P, if P_{2l} is non-empty, then P_{2l} is a relative open subset in Q_{2l}. In particular, $P_{\max}$ is an open connected density subset of P if P is a connected holomorphic Poisson manifold.

Proof. From equation (2.3) and (2.4), we know $Q_{2l} = P_{2l} \bigcup Q_{2l-2}$. Note Q_{2l} and Q_{2l-2} are closed, so P_{2l} is a relative open in Q_{2l}. Since $P_{\max}$ is the complement of the closed subset $Q_{\max-2}$ in $Q_{\max}$ and a real codimension two subset doesn't destroy the connectedness of a manifold, $P_{\max}$ is an open connected density subset of P. □

Any smooth Poisson manifold P has the rank decomposition $P = \sum P_{2l}$, where P_{2l} is foliated by symplectic leaves of dimension $2l$. There is an open subset $P_{\max}$ which is of maximal rank, but it does not necessary dense in P. (Here is one such example. The Poisson manifold $P = \mathbb{R}^2$ with Poisson tensor $\pi_P = f(x,y)\partial_x \bigwedge \partial_y$. Here f is a smooth function such that $f(x,y) = 0$ if $y \geq 0$, and $f(x,y) = x$ when $y < 0$. Note $P_{\max} = \{(x,y) \in R^2 | x \neq 0, y < 0\}$ in this example). Since Hamiltonian flows preserve the rank of the Poisson bivector field, all P_{2l} are Ham(P)-invariant subsets of P (Assume P is compact when Ham(P) is not appropriately defined). Denote the connected components of Ham(P)-orbits by $\mathcal{F}_i$, where i is indexed by an index set denoted by Λ_l. Then we have a symplectic decomposition

$$P = \sum_{\lambda \in \Lambda_l, 1 \leq l \leq [n/2]} \mathcal{F}_l^\lambda .$$

The symplectic leaves in $P_{\max}$ are all open. So the Ham(P)-action has at least one open orbit. If (P, π_P, J) is a holomorphic symplectic manifold, the Hamiltonian vector field of any holomorphic function is holomorphic. So, all symplectic leaves are complex symplectic submanifolds and the restriction of π_P to the symplectic leaf $\mathcal{F}_i$ defines a symplectic form ω_l^λ satisfying $\omega_l^\lambda(J\cdot, J\cdot) = \omega_l^\lambda(\cdot,\cdot)$. Hence $\mathcal{F}_i$ are pseudo-Kähler submanifolds. Thus we have

Proposition 1.2 Let P be a holomorphic Poisson manifold. Then we have a symplectic decomposition

$$P = \sum_{\lambda \in \Lambda_l, 1 \leq l \leq [n/2]} \mathcal{F}_l^\lambda .$$

Where each symplectic leaf $\mathcal{F}_l^\lambda$ is a pseudo-Kähler submanifold. Moreover P has at least one open symplectic leaf.

§3. Poisson Lie group

Before giving the definition of a Poisson Lie group, we need some notations. Let $\mathcal{A}, \mathcal{B}$ be Poisson algebras over $I\!F$. A homomorphism $f : \mathcal{A} \longrightarrow \mathcal{B}$ is called a *Poisson homomorphism* if it is bracket preserved, *i.e.* , $f(\{a,b\}) = \{f(a), f(b)\}$. A smooth map between smooth Poisson manifolds M and N is called a *Poisson map* if the pull back map $f^* : C^\infty(N) \longrightarrow C^\infty(M)$ is a Poisson homomorphism. Given two Poisson manifolds (M, π_M) and (N, π_N), there is a unique Poisson structure,

denoted by $\pi_M \oplus \pi_N$, on the product manifold $M \times N$, such that the projections from $M \times N$ to each factor M and N are both Poisson maps. We called it the *product Poisson structure.* For algebraic varieties (analytic spaces), we have similar definitions of Poisson morphisms and Poisson structure on Product varieties (spaces).

A Lie (resp. algebraic) group which is at the same time a Poisson manifold is called a *Poisson Lie (resp. algebraic) group* if the multiplicative map $\mu : G \times G \longrightarrow G, (h_1, h_2) \longmapsto h_1 h_2$ is a Poisson map, where $G \times G$ is equipped with the product Poisson structure.

Let r_g and l_g respectively denote the left and right translation in G by g. Then the Poisson bivector field π_G of a Poisson Lie group satisfies

$$\pi_G(gh) = l_g \pi_G(h) + r_h \pi_G(g), \forall g, h \in G,$$

and is called a *multiplicative* bivector field. If both g, h are specialized to be the unit e, then $\pi_G(e) = 0$. So a Poisson Lie groups can never be a symplectic manifold. Let $\pi^r(g) = r_{g^{-1}} \pi_G(g)$. Then

$$\pi^r(gh) = \pi^r(g) + Ad_g \pi^r(h) \tag{3.1}$$

The linearization of π^r at the unit is a $\bigwedge^2 \mathfrak{g}$-valued 1-cocycle on $\mathfrak{g}$ relative to the adjoit action of $\mathfrak{g}$ on $\wedge^2 \mathfrak{g}$, that is $\delta = d_e \pi^r \in H^1(\mathfrak{g}, \wedge^2 \mathfrak{g})$. If δ is a 1-coboundary, then G is called a *coboundary Poisson-Lie group.* The dual map δ^* defines a Lie algebra structure on the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$. There is a Lie algebra structure on $\mathfrak{g} \oplus \mathfrak{g}$ such that both $\mathfrak{g}$ and $\mathfrak{h}^*$ are Lie subalgebras. $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}, \mathfrak{g}^*)$ is called the *Lie bialgebra* of (G, π_G). There is a 1-1 correspondence between connected and simplely connected Poisson Lie groups and Lie bialgebras (cf. [LW]). The connected and simply connected group G^* with Lie algebra $\mathfrak{g}^*$ is also a Poisson Lie group, called the *dual* Poisson Lie group of G. A Lie group $D(G)$ with Lie algebra $\mathfrak{g} \oplus \mathfrak{g}^*$ is called a *double* for the Poisson Lie group G if the multiplication map $G^* \times G \longrightarrow D(G), (g, h) \longmapsto gh$ is a diffeomorphism. In this case, the left-action of $D(G)$ on itself induces an action on G^*. Its restriction $G \times G^* \longrightarrow G^*, (g, h) \longmapsto D^l(g)$ i s called the *left dressing action* of G on G^*. If G has trivial Poisson Lie structure, $G^* = \mathfrak{g}^*$ has Lie Poisson structure, and the left drssing action is just coadjoint action of G on G^*. The definition of right dressing action is similar (cf. [Lu]).

If G is semisimple, the Whitehead Theorem says that both $H^1(\mathfrak{g}, V)$ and $H^2(\mathfrak{g}, V)$ are vanishing for any $\mathfrak{g}$-module V. Thus every connected semi-simple Poisson Lie group is coboundary Poisson Lie

group. The 1-cocycle is given by $\delta(X) = ad_X\Lambda$ for some $\Lambda \in \wedge^2\mathfrak{g}$. By integration we get a 2-cycle $\pi_G(g) = l_g\Lambda - r_g\Lambda$ on the group G. Clearly π_G is a Poisson bivector if and only if $[\Lambda, \Lambda] \in (\wedge\mathfrak{g})^{\text{inv}}$. Such Λ is called a **r**-*matrix*.

If G is an abelian group, then $g \longrightarrow \pi^r(g)$ is a Lie group homomorphism from G to abelian group $\bigwedge^2\mathfrak{g}$. So on the compact abelian group there is no nontrivial Poisson Lie group structure.

If $G = T^m \times \mathbb{R}^n$, a direct product of torus group and vector group. Let

$$\overline{\pi}^r(\theta_1, \cdots, \theta_m, x_1, \cdots, x_n) = \pi^r(e^{i\theta_1}, \cdots, e^{i\theta_m}, x_1, \cdots, x_n)$$

be the lift of π^r on the universal covering space $\mathbb{R}^{m+n}$. By the multiplicative condition (3.1), we know $\overline{\pi}^r_{ij}$ is a linear function with respected to the variables θ_ρ and x_δ and periodic with respected to θ_ρ. So the Poisson bivector has the form

$$\overline{\pi}^r(u_1, \cdots, u_{m+n}) = \sum_{i,j,k=1}^{m+n} C_{ij}^k u_k \partial_{u_i} \bigwedge \partial_{u_j},$$

where the C_{ij}^k are the structure constants of a $m+n$-dimensional Lie algebra and $C_{ij}^k = 0$ for $k \leq m$; $u_i = \theta_i$ for $1 \leq i \leq m$ and $u_{m+i} = x_i$ for $1 \leq i \leq n$. In particular, if G is the vector group $\mathbb{R}^m$, the Poisson Lie structure on G is linear and called a *Lie Poisson* structure. For multiplicative abelian group $G = (\mathbb{R}_+)^n$ or $(\mathbb{C}^\star)^n$. the solution of (3.1) is of form

$$\pi^r_{\mu\nu}(z_1, \cdots, z_n) = \sum_{\delta=1}^{n} C_{\mu\nu}^\delta \ln z_\delta, \tag{3.2}$$

π_G is a Poisson structure iff $C_{\mu\nu}^\delta = -C_{\nu\mu}^\delta \in I\!\!F$ for $1 \leq \mu, \nu, \delta \leq n$ and

$$\sum_{\mu,\nu=1}^{n} [(C_{\rho\delta}^\mu C_{\delta\gamma}^\nu + C_{\rho\gamma}^\mu C_{\delta\gamma}^\nu)\ln z_\mu \ln z_\nu + C_{\rho\nu}^\mu C_{\delta\gamma}^\nu \ln z_\mu] + \text{c.p.}(\rho, \delta, \gamma) = 0 \tag{3.3}$$

for $1 \leq \rho, \delta, \gamma \leq n$, here c.p.$(\rho, \delta, \gamma)$ means cyclic permutation with respected to ρ, δ and γ. After some simple calculations, we know (3.3) is equivalent to

$$\sum_{\nu=1}^{n} C_{\rho\nu}^\mu C_{\delta\gamma}^\nu + C_{\delta\nu}^\mu C_{Z\rho}^\nu + C_{\gamma\nu}^\mu C_{\rho\delta}^\nu = 0, \quad 1 \leq \mu, \rho, \delta, \gamma \leq n. \tag{3.4}$$

So $C_{\rho\delta}^\mu$ are also the structure constants of a n-dimensional Lie algebra. The Poisson Lie group structure on G is given by

$$\pi_G(z_1, \cdots, z_n) = \sum_{\delta,\mu,\nu=1}^{n} C_{\mu\nu}^\delta z_\mu z_\nu \ln z_\delta \partial_\mu \wedge \partial_\nu$$

Example 3.3 $\pi = x(ae^{i(\theta_1+\theta_2)}\partial_{\theta_1} \wedge \partial_{\theta_2} + be^{i\theta_1}\partial_{\theta_1} \wedge \partial_x + ce^{i\theta_2}\partial_{\theta_2} \wedge \partial_x)$ is a Poisson Lie group structure on $T^2 \times \mathbb{R}$ for any $a, b, c, \theta_1, \theta_2 \in \mathbb{R}$.

Belavin and Drinfeld classified all Poisson Lie structures on complex simple Lie group [BD], the classification of Poisson-Lie structure on compact connected Lie group was carried out by Levendorskii and Soibelman in [LS]. The classification of Poisson Lie structure on non-semisimple non-abelian group is more involved. In [M], [Z], [BKM], where the cases 2-dimention affine group, Poincaré group, Galilei group, respectively, were solved completely.

§4 Poisson actions and momentum mappings

An action of a Poisson Lie group (G, π_G) on a Poisson manifold (P, π_P) is called a *Poisson action* if the action map $\sigma : G \times P \longrightarrow P$ is a Poisson map, where $G \times P$ is equipped with product Poisson structure. Let $g \in G$ and $x \in P$, denote by $\sigma_g : P \longrightarrow P$ and by $\sigma_x : G \longrightarrow P$ the maps

$$\sigma_g : x \longmapsto \sigma(g, x) = gx, \quad \sigma_x : g \longmapsto \sigma(g, x) = gx.$$

σ is a *Poisson action* iff

$$\pi_P(gx) = \sigma_{g*}\pi_P(x) + \sigma_{x*}\pi_G(g).$$

Denote the infinitesimal action by $\lambda : \mathfrak{g} \longrightarrow \Gamma(TP)$, equivalently, $\lambda(X)(x) = \sigma_{x*}X$. For any $f \in C^\infty(P)$, let $\xi_f \in \mathfrak{g}^*$ be the covector defined by

$$\langle \xi_f(p), X \rangle = \frac{d}{dt}\Big|_{t=0} f(e^{tX}p) = \langle df(p), \lambda(X) \rangle. \tag{4.1}$$

If G is connected, σ is a Poisson action iff

$$\lambda(X)(\{f, g\}) = \{\lambda(X)(f), g\} + \{f, \lambda(X)(g)\} + \langle X, [\xi_f, \xi_g]_* \rangle, \tag{4.2}$$

Clearly, if G has trivial Poisson structure, in this case $G^* = \mathfrak{g}^*$ is an abelian Poisson Lie group, the G-action preserves the Poisson structure of P. Conversely, we have

Proposition 4.1 Let $\sigma : G \times P \longrightarrow P$ be a Poisson action. Then it preserves the Poisson structure of P if and only if for every $p \in P$, the annihilator $\mathfrak{g}_p^\circ$ of the isotropy subalgebra $\mathfrak{g}_p$ at p is abelian. In particular, if the action is locally free, then σ is a Poisson structure preserved action if and only if G is a trivial Poisson Lie group.

Proof. σ preserves the Poisson structure of P if and only if

$$\lambda(X)(\{f,g\}) = \{\lambda(X)(f), g\} + \{f, \lambda(X)(g)\}, \tag{4.3}$$

Define the map $A : d(C^\infty(P)) \longrightarrow \mathfrak{g}^*, df \longmapsto \xi_f$. Then by (4.1), the image of A is the annihilator of $\mathfrak{g}_p = \{X|\lambda(X)(p) = 0\}$. By (4.2) and (4.3), we know the action preserves the Poisson structure of P if and only if for every $p \in P$, the annihilator $\mathfrak{g}_p^\circ$ of the isotropy subalgebra $\mathfrak{g}_p$ is abelian. So the linearization of Poisson structure of G is zero, which means the Poisson Lie structure of G is trivial. □

Poisson action $\sigma : G \times P \longrightarrow P$ is said to be *tangential* if every infinitesimal vector field $\lambda(X)$ is tangent to the symplectic leaf of P. A smooth map $m : P \longrightarrow G^*$, is called a *momentum map*, if $\lambda(X) = \pi^\sharp(J^* X^l)$, here X^l is the left invariant 1-form on G^* whose value at the identity is $X \in \mathfrak{g} = (\mathfrak{g}^*)^*$. A momentum map is called equivariant if it intertwines with the action of σ and the left dressing action of G on G^*. A Poisson action with momentum map is clearly tangential action. Dressing actions are tangential Poisson actions (cf. [STS, LW]). The infinitesimal vector field of left dressing action of G on G^* is given by

$$\mathrm{d}^\mathrm{l}(\mathrm{X}) = \pi^\sharp_{\mathrm{G}^*}(\mathrm{X}^\mathrm{l}).$$

Clearly by the definition of dressing action, the dressing orbits sweep out all symplectic leaves of G^* and the identity map is an equivariant moment map.

In general case if there exists a momentum map, it is not necessary equivariant. Let m be a momentum map which is not necessary eqivariant, define a map

$$\Sigma : G \times P \longrightarrow G^*, (g, x) \longmapsto (D^l(g)m(x))^{-1} \cdot m(gx).$$

Σ measures the equivariant properties of m, it is equivariant if and only if the image of Σ is a single point. We want to give an infinitesimal description of Σ. For that we fix $x \in P$, denote $\Sigma^x : G \longrightarrow G^*, g \longmapsto \Sigma(g, x)$ and $\Gamma_{X,Y}(x) = \langle d_e\Sigma^x(X), Y\rangle$, where $X, Y \in \mathfrak{g}$. Then we have

Proposition 4.2 (i)

$$\Gamma_{X,Y} = m^*(\pi_{G^*}(X^l, Y^l)) - \pi_P(m^* X^l, m^* Y^l); \qquad (4.4).$$

(ii)

$$\pi^\sharp_P(d\Gamma_{X,Y}) = \pi^\sharp_P(m^*(i(\mathrm{d}^\mathrm{l}(\mathrm{X}))\mathrm{dY}^\mathrm{l}) - \mathrm{i}(\mathrm{d}^\mathrm{l}(\mathrm{Y}))\mathrm{dX}^\mathrm{l}) - \mathrm{i}(\lambda(\mathrm{X}))\mathrm{dm}^*\mathrm{Y}^\mathrm{l} + \mathrm{i}(\lambda(\mathrm{Y}))\mathrm{dm}^*\mathrm{X}^\mathrm{l}) \qquad (4.5).$$

(iii) For $Z \in \mathfrak{g}$, the differential of $\Gamma_{X,Y}$ in the direction of $\lambda(Z)$ is

$$\begin{aligned} \lambda(Z)(\Gamma_{X,Y}) &= \langle \lambda(Z), m^*(\{X^l, Y^l\}_{\pi_{G^*}}) - \{m^* X^l, m^* Y^l\}_{\pi_P} \rangle \\ &\quad + dX^l(m_*(\lambda(Y)) - \mathrm{d^l(Y), m_*(\lambda(Z))}) \\ &\quad + dY^l(\mathrm{d^l(X) - m_*(\lambda(X)), m_*(\lambda(Z))}) \end{aligned} \tag{4.6}$$

(iv) Define the map $\Gamma : \mathfrak{g} \times \mathfrak{g} \longrightarrow C^\infty(P), (X,Y) \longmapsto \Gamma_{X,Y}$. Then Γ is antisymmetric and bilinear and

$$\begin{aligned} & (d_*\Gamma + \tfrac{1}{2}[\Gamma,\Gamma])(X,Y,Z) \\ = & \{\Gamma([X,Y],Z) - \langle Z, [(m_*(\lambda(X)))^l - (\mathrm{d^l(X))^l, (m_*(\lambda(Y)))^l - (d^l(Y))^l}]_* \rangle\} \\ & +\text{c.p.}(X,Y,Z) \\ = & \{\Gamma([X,Y],Z) - [(L_{\lambda(Z)}\pi_P)(m^*X^l, m^*Y^l) + (L_{\mathrm{d^l(Z)}}\pi_{G^*})(X^l, Y^l)] \\ & +[(L_{\overline{Z}}\pi_G)(e)((\mathrm{d^l(X))^l, (m_*(\lambda(Y)))^l}) + (\mathrm{L}_{\overline{Z}}\pi_{\mathrm{G}})(\mathrm{e})((\mathrm{m_*(\lambda(X)))^l, (d^l(Y))^l})]\} \\ & +\text{c.p.}(X,Y,Z); \end{aligned} \tag{4.7}$$

here $d : \Gamma(\bigwedge^* T^*P) \longrightarrow \Gamma(\bigwedge^{*+1} T^*P)$ is usual de Rahm differential on forms decided by the Lie brackets of vector fields on P; and $d_* : \mathrm{Hom}(\bigwedge^* \mathfrak{g}, C^\infty(P)) \longrightarrow \mathrm{Hom}(\bigwedge^{*+1} \mathfrak{g}, C^\infty(P))$ is the de Rham differential decided by the Lie bracket of $\mathfrak{g}^*$ which we denoted by $[,]_*$; and $\overline{Z}$ is any vector field on G whose value at the unit of G is Z.

Proof. (i) Σ^x maps the unit of G to that of G^*, so there is an induced map $d_e\Sigma^x : \mathfrak{g} \longrightarrow \mathfrak{g}^*$. And,

$$\begin{aligned} \Gamma_{X,Y} &= \langle d_e\Sigma^x(X), Y \rangle \\ &= \langle (l_{m(x)^{-1}})_*(m_*(\lambda(X))(x)) - (l_{m(x)^{-1}})_*(\mathrm{d^l(X)(m(x)), Y}) \rangle \\ &= \langle m_* X_P(x) - \mathrm{d^l(X)(m(x)), Y^l} \rangle \\ &= \langle m_* X_P, Y^l \rangle + \langle \pi^\sharp_{G^*}(X^l) \circ m, Y^l \rangle \\ &= \langle \pi^\sharp_{G^*}(X^l), Y^l \rangle \circ m - \langle m_*(\pi^\sharp_P(m^* X^l)), Y^l \rangle \\ &= m^*\pi_{G^*}(X^l, Y^l) - \pi_P(m^* X^l, m^* Y^l). \end{aligned}$$

(ii) Using the defintion (2.1), we have,

$$\begin{aligned} d\Gamma_{X,Y} &= m^*(\{X^l, Y^l\}_{\pi_{G^*}} - i(\pi^\sharp_{G^*}(X^l))Y^l + i(\pi^\sharp_{G^*}(Y^l))X^l) \\ &\quad -\{m^*X^l, m^*Y^l\}_{\pi_P} + i(\pi^\sharp_P(m^*X^l))dm^*(Y^l) - i(\pi^\sharp_P(m^*Y^l))dm^*(X^l) \\ &= m^*(\{X^l, Y^l\}_{\pi_{G^*}}) - \{m^*X^l, m^*Y^l\}_{\pi_P} \\ &\quad -i(\lambda(X))dm^*(Y^l) + i(\lambda(Y))dm^*(X^l) - m^*(-i(\mathrm{d^l(X)})dY^l + i(\mathrm{d^l(Y)})dX^l) \end{aligned}$$

Using the following identity

$$\begin{aligned} & \pi_P^\sharp(m^*(\{X^l,Y^l\}_{\pi_{G^*}})-\pi_P^\sharp\{m^*X^l,m^*Y^l\}_{\pi_P}) \\ = \; & \pi_P^\sharp(m^*([X,Y]^l)-[\pi_P^\sharp(m^*X^l),\pi_P^\sharp(m^*Y^l)] \\ = \; & \lambda([X,Y])+[\lambda(X),\lambda(Y)]=0 \end{aligned}$$

to cancel the first two terms in the expression of $d\Gamma_{X,Y}$, then applying map $\pi^\sharp$ to it we get (4.5).

(iii) Firt note that $\lambda(Z)(\Gamma_{X,Y})=\langle\lambda(Z),d\Gamma_{X,Y}\rangle$, since

$$\begin{aligned} & i(\lambda(Z))(m^*(i(\mathrm{d^l(X))dY^l})-\mathrm{i(\lambda(X))dm^*(Y^l)+i(\lambda(Y))dm^*(X^l)-m^*(i(d^l(Y))dX^l)}) \\ = \quad & dX^l(m_*(\lambda(Y))-\mathrm{d^l(Y),m_*(\lambda(Z)))+dY^l(d^l(X)-m_*(\lambda(X)),m_*(\lambda(Z)))}, \end{aligned}$$

and the calculations in (ii) we can easily get (4.6).

(iv)Using

$$\frac{1}{2}[\Gamma,\Gamma](X,Y,Z)=\langle Z,\Gamma[X,Y]_\Gamma-[\Gamma X,\Gamma Y]_*\rangle,$$

and

$$\langle Z,\Gamma[X,Y]_\Gamma\rangle=-\langle\Gamma Z,ad^*_{\Gamma X}Y-ad^*_{\Gamma Y}X\rangle=-\langle Y,[\Gamma Z,\Gamma X]_*\rangle-\langle X,[\Gamma Y,\Gamma Z]_*\rangle,$$

we know

$$\begin{aligned} & (d_*\Gamma+\tfrac{1}{2}[\Gamma,\Gamma])(X,Y,Z) \\ & =\langle Z,\Gamma[X,Y]-[\Gamma X,\Gamma Y]_*\rangle+\mathrm{c.p.}(X,Y,Z) \\ & =\langle Z,(m_*[X,Y]_P)^l-([X,Y]_{G^*})^l\rangle-[(m_*(\lambda(X)))^l-\mathrm{(d^l(X))^l,(m_*(\lambda(Y)))^l-(Y_{G_*})^l]_*\rangle} \\ & \quad +\mathrm{c.p.}(X,Y,Z) \\ & =\{\Gamma_{[X,Y],Z}-\langle Z,[(m_*(\lambda(X)))^l-\mathrm{(d^l(X))^l,(m_*(\lambda(Y)))^l-(Y_{G_*})^l]_*\rangle\}} \\ & \quad +\mathrm{c.p.}(X,Y,Z). \end{aligned}$$

Since $G\times P\longrightarrow P$ and $G\times G^*\longrightarrow G^*$ are Poisson actions, by Theorem 4.7 of [Lu],

$$(L_{\lambda(Z)}\pi_P)(m^*X^l,m^*Y^l)=\langle Z,[(m_*(\lambda(X)))^l,(m_*(\lambda(Y)))^l]_*\rangle; \tag{4.8}$$

$$(L_{\mathrm{d^l(Z)}}\pi_{G^*})(X^l,Y^l)=\langle Z,[\mathrm{(d^l(X))^l,(d^l(Y))^l}]_*\rangle; \tag{4.9}$$

And by the definition of $[,]_*$, we have,

$$(L_{\overline{Z}}\pi_G)(e)(\mathrm{(d^l(X))^l,(m_*(\lambda(Y)))^l})=\langle\mathrm{Z},[\mathrm{(d^l(X))^l,(m_*(\lambda(Y)))^l}]_*\rangle; \tag{4.10}$$

$$(L_{\overline{Z}}\pi_G)(e)((\mathrm{m}_*(\lambda(X)))^l,\mathrm{(d^l(Y))^l})=\langle\mathrm{Z},[\mathrm{m}_*(\lambda(\mathrm{X})))^l,\mathrm{(d^l(Y))^l}]_*\rangle. \tag{4.11}$$

By (4.8-4.11), we have the last identity of (4.7). $\square$

If G is connected, the momentum map $m : P \longrightarrow G^*$ is equivariant if and only if m is a Poisson map by (ii) of Proposition 4.2. If P is a symplectic manifold, and $x_0 \in P$ such that $m(x_0) = e^*$ is the unit of G^*. Let $\Lambda = m_*\pi_{x_0} \in \bigwedge^2 \mathfrak{g}^*$, and $\pi_{\rm tw} = \pi_{G^*} + \Lambda^r$. Then $\pi_{\rm tw}$ is still a Poisson structure of G^*. It is proved in [Lu] that $m : (P, \pi_P) \longrightarrow (G^*, \pi_{\rm tw})$ is an equivariant momentum map. Which means $m^*((\pi_{G^*} + \Lambda^r)(X^l, Y^l)) - \pi_P(m^*X^l, m^*Y^l) = 0$. So

$$\Gamma(X, Y)(x) = (Ad(m(x)^{-1})\Lambda)(X, Y).$$

Since $\pi_{\rm tw}$ is a Poisson structure, we have $d_*\Lambda + \frac{1}{2}[\Lambda, \Lambda] = 0$. But we even don't know when $d_*\Gamma + \frac{1}{2}[\Gamma, \Gamma] = 0$ on the symplectic leaves of P.

From now on we suppose G is a trivial Poisson Lie group. Then $\pi_G = 0$ and $G^* = \mathfrak{g}^*$. Denote the set of Carsimir functions on P by $\mathrm{Car}(P)$. In this case $\Gamma_{X,Y} \in \mathrm{Car}(P)$ is a Carsimir function and

$$\Gamma_{X,Y}(x) = m([X, Y])(x) - \{m(X), m(Y)\}_{\pi_P}(x),$$

where we denote the X-component of $m(x)$ by $m(X) \in C^\infty(P)$. By definition the Poisson structure of P is preserved under the action of G we have $L_{\lambda(Z)}\pi_P = L_{\mathrm{d^l(Z)}}\pi_{G^*} = L_{\overline{Z}}\pi_G = 0$. So $\frac{1}{2}[\Gamma, \Gamma](X, Y, Z) = 0$ and (4.7) is reduced to

$$\begin{aligned}
& d_*\Gamma(X, Y, Z) \\
= \; & (\{m([X, Y], m(Z)\}_{\pi_P} - m([[X, Y], Z])) + \text{c.p.}(X, Y, Z) \\
= \; & (\{\{m(X), m(Y)\}_{\pi_P} - \Gamma_{X,Y}, m(Z)\}_{\pi_P} - m([[X, Y], Z])) + \text{c.p.}(X, Y, Z) \\
= \; & (\{\{m(X), m(Y)\}_{\pi_P}, m(Z)\}_{\pi_P} - m([[X, Y], Z])) + \text{c.p.}(X, Y, Z) \\
= \; & 0.
\end{aligned}$$

Hence $\Gamma : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathrm{Car}(P)$ is a $\mathrm{Car}(P)$-valued 2-cocycle of $\mathfrak{g}$ relative to the trivial representation of $\mathfrak{g}$ on $\mathrm{Car}(P)$, equivalently $[\Gamma] \in H^2(\mathfrak{g}, \mathrm{Car}(P)) \cong H^2(\mathfrak{g}) \otimes \mathrm{Car}(P)$. In the case that $H^1(\mathfrak{g}, \mathrm{Car}(P)) = H^2(\mathfrak{g}, \mathrm{Car}(P)) = 0$, in particular, if $\mathfrak{g}$ is semisimple, this action has a coadjoit equivariant momentum map.

Let $\Gamma_{g,Y}(x) = \langle \Sigma(g, x), Y \rangle$. If G is connected then $\Gamma_{g,Y}$ is also a Carsimir function on P. Let $L(\mathfrak{g}, \mathrm{Car}(P)) \cong \mathfrak{g}^* \otimes \mathrm{Car}(P)$ denote the linear maps from $\mathfrak{g}$ to $\mathrm{Car}(P)$. Define a map Ψ from G to

$L(\mathfrak{g}, \mathrm{Car}(P))$ by $\Psi(g)(Y) = \Gamma_{g,Y}$. Then

$$(\Psi(gh)(Y))(x) = (\Psi(g)(Y))(hx) + (\Psi(h)(Ad_{g^{-1}}Y))(x)$$

Because the action is tangential, hx and x lie in the same symplectic leaf and $\Gamma_{g,Y}$ is a Carsimir function, we have $(\Psi(g)(Y))(hx) = (\Psi(g)(Y))(x)$. Hence $\Psi(gh) = \Psi(g) + Ad^*_{g^{-1}}\Psi(h)$, and Ψ is a $L(\mathfrak{g}, \mathrm{Car}(P))$-value 1-cocycle on group G, it represents a cohomology class in $H^1(G, L(\mathfrak{g}, \mathrm{Car}(P))$. So G admits a coadjoint equivariant momentum map if and only if Ψ is cohomologeous to zero. In particular, if G is compact, then it admits an equivariant momentum map.

§5. Almost homogeneous tangential Poisson actions

Let G be a Lie group and P a G-manifold. P is called *G-almost homogeneous* if G has only one open orbit in P. Note here our definition of almost homogeneous is a little different to Huckleberry's in [HO] where it is defined for complex space. However ours definition can apply to real case. If G is a Poisson Li group and P is a G-almost homogeneous Poisson G space, then P is an *almost homogeneous Poisson G-manifold.* If P is an almost homogeneous Poisson manifold under the tangential action of G, then each open G-orbit must be a symplectic leaf of P, since the $\mathfrak{g}$-vector fields, which at the same time are Hamiltonian, generate the whole tangential space. So, if an almost homogeneous Poisson G-action has an equivariant momentum map, then all open G-orbits in P are symplectic covering spaces of left dressing orbits in G^*. In particular, if P is homogeneous, $P = \mathcal{O}_p = G \cdot p = G/G_p$ for some $p \in P$ is a symplectic covering space of the G-dressing orbit $\mathcal{O}_u = G/G_u \subset G^*$, here $u = m(p)$ and $m : \mathcal{O}_p \longrightarrow \mathcal{O}_u$ is a homogeneous fibration whose fibers are isomorphic to G_u/G_p.

Proposition 5.1 Let $\sigma : G \times P \longrightarrow P$ be a Poisson action on a symplectic manifold P and $\mathrm{m} : P \longrightarrow G^*$ an equivariant momentum map. Then

$$\mathrm{Ker}(\mathrm{m}_{\mathrm{p}*}) = (T_p\mathcal{O}_p)^{\omega}; \qquad \mathrm{Im}(\mathrm{m}_{\mathrm{p}*}) = [(\mathfrak{g}^l_p)]^{\circ}, \tag{5.1}$$

here the upscript ω denotes the annihilator operation with respect to symplectic structure of P.

Proof. $\forall f \in C^\infty(P)$,

$$\langle(\lambda(X))(p), df\rangle = \langle\pi^\sharp_P(m^*X^l), df\rangle = -\langle m^*X^l, \pi^\sharp_P(df)\rangle = -\langle X^l, m_{x*}(\pi^\sharp_P(df)\rangle, \tag{5.2}$$

so

$$\omega((\lambda(X))(p), \pi_P^\sharp(df)) = \langle X^l, m_*(\pi_P^\sharp(df))\rangle, \tag{5.3}$$

Let $v = \pi_P^\sharp(df) \in T_pP$, then v runs through T_pP since P is a symplectic manifold,

$$\omega((\lambda(X))(p), v) = \langle X^l, m_* v\rangle, \forall v \in T_pP.$$

So $\mathrm{Ker}(\mathrm{m_{p*}}) = \{(\lambda(X))(p) | X \in \mathfrak{g}\}^\omega = (T_x\mathcal{O}_p)^\omega$, and $(\mathrm{Im}(\mathrm{m_{p*}}))^\circ = \{X^l | (\lambda(X))(p) = 0\} = \{X^l | X \in \mathfrak{g}_p\}$, the later means $\mathrm{Im}(\mathrm{m_{p*}}) = [(L_u^{-1})^*(\mathfrak{g}_p)]^\circ$. □

Now assume G is a compact Poisson Lie group. Let

$$P_{\max} = \{x \in m(P) \mid \mathrm{dim}G_{m(x)} \leq \mathrm{dim}G_{m(y)}, \forall y \in P\}.$$

Lemma 5.1 The set $P_{\max}$ is open in P.

Proof. Since G is a compact Lie group, the action $G \times G^* \longrightarrow G^*$ is a proper action. Thus we can apply the slice theorem, it follows that for any $u \in G^*$ there is a G-invariant neighborhood $U \ni u$ with $\mathrm{dim}G_u \leq \mathrm{dim}G_v$ for all $v \in U$. The result follows from the equivariance and continuity of m. □

Proposition 5.2 For any $p \in P_{\max}$, we have $[G_u, G_u] \subset G_p$, where $u = m(p)$.

Proof. Let $\gamma : [0, \epsilon] \longrightarrow P_{\max}$ be a smooth curve with $\gamma(0) = p$ and $\gamma'(0) = V \in T_pP$. Note that $\mathrm{dim}\mathfrak{g}_{m(\gamma(t))}$ is a locally constant near $\gamma(t)$, hence for any $X, Y \in \mathfrak{g}_u$, we can find smooth curves X_t, $Y_t \in \mathfrak{g}_{m(\gamma(t))}$. So

$$\pi_{G^*}(m(\gamma(t)))(X_t^l, Y_t^l) = \langle \pi_{G^*}^\sharp(X_t^l), Y_t^l\rangle(m(\gamma(t)) = \langle d^l(X)(m(\gamma(t)), Y_t^l\rangle = 0,$$

Thus

$$\pi_{G^*}^l(m(\gamma(t)))(X_t, Y_t) = 0. \tag{5.4}$$

Multiplying $l_{(m(\gamma(t)))^{-1}}$ with both side of (5.4) then differentiating it at $t = 0$, we get $(L_{(m_*V)}\pi_{G^*})(X_t^l, Y_t^l) = 0$. Use (2.2) and the fact that $\{X_t^l . Y_t^l\} = [X_t, Y_t]^l$, we have

$$\langle m_*V, [X_t, Y_t]^l\rangle = \langle \pi_{G^*}^\sharp(X_t^l), d(i(m_*V)Y_t^l)\rangle - \langle \pi_{G^*}^\sharp(Y_t^l), d(i(m_*V)X_t^l)\rangle = 0.$$

Varying $V \in T_pP$, we have $[X, Y]^l \in \mathrm{Im}(m_*)$. Use Proposition 5.1, we immediately have $[\mathfrak{g}_u, \mathfrak{g}_u] \subset \mathfrak{g}_p$. □

Theorem 5.1 Let G be a compact Poisson Lie group and P an almost homogeneous Poisson G-manifold with an equivariant momentum map $\mathrm{m} : P \longrightarrow G^*$. Then the open G-orbit in P is a symplectic fibre bundle over a dressing orbit. In particular, a compact homogeneous Poisson manifold with an equivariant momentum map is a symplectic torus bundle over a dressing orbit.

Proof. The first part of this theorem follows from the discussions at the beginning of this section. If P is a compact homogeneous Poisson manifold with an equivariant momentum map m, let $P = \mathcal{O}_p = G \cdot p = G/G_p$ for some $p \in P$ and $\mathcal{O}_u = G/G_u \subset G^*$, here $u = m(p)$. Then P is a symplectic homogeneous fibre bundle over the dressing orbit $\mathcal{O}_u$. Since P is compact, the fibre is isomorphic to G_u/G_p and is compact as well. By Proposition 5.2, P is a torus bundle over $\mathcal{O}_u$. □

At the end of this section we give an example of non-transitive almost homogeneous Poisson action.

Example 5.1 Consider the Lie group

$$SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R}, ad - bc = 1 \right\}.$$

We denote

$$e_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_3 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

a basis of its Lie algebra $sl(2,\mathbb{R})$. Then

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = -e_2.$$

Let $\Lambda = \lambda_1 e_1 \wedge e_2 + \lambda_2 e_2 \wedge e_3 + \lambda_3 e_3 \wedge e_1$, where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. Clearly it is a $\mathbf{r}$-matrix of $sl(2,\mathbb{R})$. Let $\pi(g) = L_{g*}\Lambda - R_{g*}\Lambda$ with $g \in SL(2,\mathbb{R})$. Then $(SL(2,\mathbb{R}), \pi)$ is a Poisson Lie group. The dual Lie bracket are defined by $[\xi, \eta]_* = ad^*_{\Lambda\xi}\eta - ad^*_{\Lambda\eta}\xi$ for any $\xi, \eta \in sl^*(2,\mathbb{R})$. Let e_1^*, e_2^*, e_3^* be the basis of $sl^*(2,\mathbb{R})$ dual to e_1, e_2, e_3. It is easy to check the Lie brackets on this basis are given by

$$[e_1^*, e_2^*]_* = -\lambda_3 e_1^* - \lambda_2 e_2^*, \quad [e_2^*, e_3^*]_* = \lambda_1 e_2^* - \lambda_3 e_3^*, \quad [e_3^*, e_1^*]_* = \lambda_2 e_3^* - \lambda_1 e_1^*.$$

Let $\pi_{\mathbb{R}^2} = h(x_1, x_2)\partial_{x_1} \wedge \partial_{x_1}$. Then $(\mathbb{R}^2, \pi_{\mathbb{R}^2})$ is a Poisson manifold for any $h(x_1, x_2) \in C^\infty(\mathbb{R}^2)$. The natural action of $SL(2,\mathbb{R})$ on $\mathbb{R}^2$ is not transitive and has two orbits: a fixed point orbit $\mathcal{O}_1 = \{0\}$ and an open orbit $\mathcal{O}_2 = \mathbb{R}^2 - \{0\}$. In order that the natural action $\sigma : SL(2,\mathbb{R}) \times (\mathbb{R}^2, \pi_{\mathbb{R}^2}) \longrightarrow$

$(\mathbb{R}^2, \pi_{\mathbb{R}^2})$ is a Poisson action, we must have

$$\pi_{\mathbb{R}^2}(gx) = \sigma_{g*}\pi_{\mathbb{R}^2}(x) + \sigma^x_*\pi(g). \tag{5.5}$$

Let

$$g = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in SL(2, \mathbb{R}), \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2,$$

Then (5.5) can be written as

$$\begin{aligned} & h(a_1x_1 + a_2x_2, a_3x_1 + a_4x_2) - h(x_1, x_2) \\ = \ & \tfrac{1}{4}\{[(\lambda_1 + \lambda_3)a_1^2 - (\lambda_1 - \lambda_3)a_3^2 - 2\lambda_2 a_1 a_3 - (\lambda_1 + \lambda_3)]x_1^2 \\ & +[(\lambda_1 + \lambda_3)a_2^2 - (\lambda_1 - \lambda_3)a_4^2 - 2\lambda_2 a_2 a_4 + (\lambda_1 - \lambda_3)]x_2^2 \\ & +[2(\lambda_1 + \lambda_3)a_1a_2 - 2(\lambda_1 - \lambda_3)a_3a_4 - 4\lambda_2 a_2 a_3]x_1x_2\} \end{aligned} \tag{5.6}$$

The solution of (5.6) is

$$h(x_1, x_2) = \frac{1}{4}(\lambda_1 + \lambda_3)x_1^2 - \frac{1}{4}(\lambda_1 - \lambda_3)x_2^2 - \frac{1}{2}\lambda_2 x_1 x_2 + c, \qquad c \in \mathbb{R}.$$

The infinitesimal action of the natural action of $SL(2, \mathbb{R})$ on $\mathbb{R}^2$ is

$$(e_1)_P = \frac{1}{2}(x_1\partial_{x_1} - x_2\partial_{x_2}), \quad (e_2)_P = \frac{1}{2}(x_2\partial_{x_1} - x_1\partial_{x_2}), \quad (e_3)_P = \frac{1}{2}(x_2\partial_{x_1} + x_1\partial_{x_2}).$$

So this action is a tangential Poisson action if and only if $\lambda_1, \lambda_2, \lambda_3, c$ satisfy

$$\lambda_1 + \lambda_3 > 0, \quad {\lambda_1}^2 + {\lambda_2}^2 - {\lambda_3}^2 < 0, \quad c \geq 0. \tag{5.7}$$

For example, if $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 4$, then $h(x_1, x_2) = x_1^2 + x_2^2 + c$. The natural action $SL(2, \mathbb{R}) \times (\mathbb{R}^2, \pi_{\mathbb{R}^2}) \longrightarrow (\mathbb{R}^2, \pi_{\mathbb{R}^2})$ is a tangential Poisson action if $c \geq 0$.

In the following, we consider the case $\lambda_1 = \lambda_3 = 0$ and $\lambda_2 = 2$, and take $h(x_1, x_2) = c - x_1x_2$. The dual Poisson Lie group $SL(2, \mathbb{R})^*$ is realized explicitly by [Lu]:

$$SL^*(2, \mathbb{R}) \overset{\text{Def}}{=} SB(2, \mathbb{R}) \overset{\text{Def}}{=} \left\{ \begin{pmatrix} a & b + ic \\ 0 & a^{-1} \end{pmatrix} : a > 0, b, c \in \mathbb{R}, \right\}.$$

The dual basis is realized by

$$e_1^* = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_2^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_3^* = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}.$$

And the dual Lie brackets are given by

$$[e_1^*, e_2^*]_* = -2e_2^*, \quad [e_2^*, e_3^*]_* = 0, \quad [e_3^*, e_1^*]_* = 2e_3^*.$$

Noticing that $(e_1)^\circ \cong \mathrm{Span}_{\mathbb{R}}\{e_2, e_3\}$ is an ideal of $sb(2, \mathbb{R})$, so $H = \mathrm{Diag}(a, a^{-1})$ is a Poisson Lie subgroup of $SL(2, \mathbb{R})$ (cf. [Y]). Each orbit of the restricted action of H on $\mathbb{R}^2$ if of form $\{(x, y) \in \mathbb{R}^2 : xy = c\}$, where c is any constant. Clearly this action is Poisson structure preserved and tangential, it has a family of momentum maps:

$$\mathrm{m}_H : \mathbb{R}^2 \longrightarrow \mathbb{R}, (x_1, x_2) \longmapsto \alpha(|c - x_1x_2|)^{-\frac{1}{2}}, \quad \alpha \in \mathbb{R}.$$

Clearly they are equivariant momentum maps.

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